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Stability of plane Poiseuille flow to periodic disturbances of finite amplitude in the vicinity of the neutral curve

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Stuart (1960) has developed a theory of the stability of plane Poiseuille flow to periodic disturbances of finite amplitude which, in the neighbourhood of the neutral curve, leads to an equation of the Landau (1944) type for the amplitude A of the disturbance: $d|A|^2/dt = k_1|A|^2 - k_2|A|^4.$
If k_2 is positive in the supercritical region ($R > R_c$) where k_1 is positive, then, according to Stuart, there is a possibility of the existence of periodic solutions of finite amplitude which asymptotically approach a constant value of $(k_1/k_2)^{\frac{1}{2}}$. We have evaluated the coefficient k_2 and found that there indeed exists a zone in the (α, R)-plane where it is positive. This is the zone inside the dashed curve shown in figure 1, with the region of instability predicted by the linear theory included inside the 'neutral curve'. Stuart's theory and Eckhaus's generalization thereof could apply in the overlapping zone just above the lower branch of the neutral curve.

1. Introduction

In his analysis of the stability of plane Poiseuille flow to periodic disturbances of finite amplitude, Stuart (1958, 1960) shows that the conjecture of Landau (1944), that the square of the amplitude ($|A|^2$) of a finite disturbance is governed by an equation of the type

$$d|A|^2/dt = k_1|A|^2 - k_2|A|^4, \tag{1}$$

can be derived on the basis of the Navier–Stokes equations. According to Stuart, (1) is an approximation valid in a region in the (α, R)-plane which is close to the 'neutral curve'. Here, α denotes the wave-number and R the Reynolds number. This work was extended by Watson (1960) and by Eckhaus (1965), and a detailed exposition of the theory is given in the latter's memoir. The constant k_1 is related to the exponential amplification-factor $\exp(\frac{1}{2}k_1 t)$ of the linear theory, as given by the Orr–Sommerfeld equation. In order to determine the constant k_2 (real) one has to solve a system of coupled ordinary differential equations, which so far has not been carried out.

An evaluation of k_2 is of interest, because the nature of the solution of (1) depends on its sign. If $k_2 > 0$, then for $R > R_c$, when $k_1 > 0$, there evidently exists a solution for which

$$d|A|^2/dt \to 0, \quad (t \to \infty), \tag{2}$$

$$|A|^2 \to (k_1/k_2). \tag{3}$$

This means that the amplitude of a periodic disturbance, which according to the linear theory would be expected to grow exponentially, actually approaches a constant value due to finite-amplitude effects. For $k_2 < 0$ and $R < R_c$, when $k_1 < 0$, equation (1) shows that $d|A|^2/dt$ can vanish for a particular value of the amplitude given by (3). This implies that the neutral curve is shifted to smaller Reynolds numbers by an amount depending on the magnitude of the disturbance. Such an effect was discussed by Meksyn & Stuart (1951).

2. Theory of a periodic disturbance of finite amplitude

We shall recapitulate here the principal steps in Stuart's analysis, and shall follow the exposition given by Eckhaus (1965). For the plane Poiseuille flow, where the laminar velocities $(\bar{u}, \bar{v})$ are given by

$$\bar{u} = 1 - y^2 \quad (-1 < y < 1), \qquad \bar{v} = 0, \tag{4}$$

let the perturbation in the velocities (u, v) be derived from a stream-function ψ

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x. \tag{5}$$

We assume that the perturbation is periodic in x:

$$\psi = \sum_{n=-\infty}^{\infty} f_n(y,t)\, e^{-i\alpha n x}, \quad f_{-n}(y,t) = \bar{f}_n(y,t), \tag{6}$$

the bar denoting the complex conjugate. The disturbance has a period L in x given by

$$L = 2\pi/\alpha, \tag{7}$$

α denoting the wave-number. Substitution of (6) in the Navier–Stokes equation

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} = \frac{1}{R}\nabla^4 \psi, \tag{8}$$

leads to a system of coupled ordinary differential equations for the f_n. Stuart's asymptotic analysis applies to a zone in the (α, R)-plane which is close to the neutral curve of the linear theory. In the linear case, the time-dependence of the functions $f_n(y,t)$ is given by a factor

$$e^{i\alpha c t}, \quad c = c_r + i c_i. \tag{9}$$

We define, with Eckhaus, a small parameter ϵ by

$$\epsilon^2 = \alpha |c_i|. \tag{10}$$

Smallness of ϵ therefore implies proximity to the neutral curve, on which c_i vanishes. It was shown by Stuart, that in an approximation in which only terms of order $\epsilon^{\frac{3}{2}}$ are retained, only the f_0, f_1 and f_2 in (6) need to be considered. These satisfy the following differential equations:

$$f_0^{\mathrm{IV}} - R\frac{\partial}{\partial t}\ddot{f}_0 = 2\alpha R \mathscr{I} \frac{\partial^2}{\partial y^2}(\bar{f}_1 \dot{f}_1 + 2\bar{f}_2 \dot{f}_2), \tag{11}$$

$$\begin{aligned} \ddot{g}_1 - \alpha^2 g_1 + i\alpha R[(1 - y^2 + \dot{f}_0) g_1 + (2 - \dddot{f}_0) f_1] - R\frac{\partial}{\partial t} g_1 \\ = i\alpha R(-\bar{f}_1 \dot{g}_2 - 2\dot{\bar{f}}_1 g_2 + \dot{f}_2 \bar{g}_1 + 2 f_2 \dot{\bar{g}}_1), \end{aligned} \tag{12}$$

$$\ddot{g}_2 - 4\alpha^2 g_2 + 2i\alpha R[(1 - y^2 + \dot{f}_0) g_2 + (2 - \dddot{f}_0) f_2] - R\frac{\partial}{\partial t} g_2 = i\alpha R(f_1 \dddot{\bar{f}}_1 - \dot{f}_1 \ddot{\bar{f}}_1). \tag{13}$$

Here, $\mathscr{I}$ denotes the imaginary part, dots signify differentiation with respect to y, and

$$g_1 = \ddot{f}_1 - \alpha^2 f_1, \quad g_2 = \ddot{f}_2 - 4\alpha^2 f_2. \tag{14}$$

As for (11), which gives the modification of the mean flow by the disturbance, we take as a first integral

$$\ddot{f}_0 - R\frac{\partial}{\partial t}\dot{f}_0 = 2\alpha R\mathscr{I}\frac{\partial}{\partial y}(\bar{f}_1\dot{f}_1 + 2\bar{f}_2\dot{f}_2). \tag{15}$$

In putting the constant of integration equal to zero in (15), the assumption was made (Watson 1960) that the mean pressure-gradient remains unchanged by the perturbation.

The solution of the system of equations (11), (12) and (13) is effected by Eckhaus through expansion in terms of the eigenfunctions $\phi_n(y)$ of the linear Orr–Sommerfeld equation:

$$f_1(y,t) = \epsilon\sum_{n=1}^{\infty} A_n(t)\,\phi_n(y). \tag{16}$$

In particular, the eigenfunction $\phi_1(y)$ of the first mode, which has the neutral curve, satisfies the equation

$$\phi_1^{\mathrm{IV}} - 2\alpha^2\ddot{\phi}_1 + \alpha^4\phi_1 + i\alpha R[(1-y^2-c_1)(\ddot{\phi}_1 - \alpha^2\phi_1) + 2\phi_1] = 0, \tag{17}$$

and is subject to the boundary conditions

$$\phi_1(1) = \dot{\phi}_1(1) = \phi_1(-1) = \dot{\phi}_1(-1) = 0. \tag{18}$$

Stuart's asymptotic analysis, valid for the zone of small ϵ, leads to the following representations:

$$f_0(y,t) = \epsilon^2|A_1(t)|^2 G_0(y) + O(\epsilon^4), \tag{19}$$

$$f_2(y,t) = \epsilon^2 A_1(t)^2 G_2(y) + O(\epsilon^4), \tag{20}$$

and $$f_1(y,t) = \epsilon A_1(t)\,\phi_1(y) + O(\epsilon^3), \qquad A_n(t) = O(\epsilon^2) \quad \text{for} \quad n > 1. \tag{21}$$

It is also argued that the term $R\partial\dot{f}_0/\partial t$ in (15) may be neglected, leading to the integral

$$\dot{G}_0(y) = -2\alpha R\mathscr{I}\int_y^1 \bar{\phi}_1\dot{\phi}_1\,dy, \tag{22}$$

and $$\ddot{G}_0(y) = 2\alpha R\mathscr{I}(\bar{\phi}_1\dot{\phi}_1). \tag{23}$$

With $\phi_1(y)$ known as the even solution of (17), $\dot{G}_0$ and $\ddot{G}_0$ are thus even functions of y which can be evaluated from (22) and (23).

Substitution of the approximations (19) to (23) in (13) gives

$$G_2^{\mathrm{IV}} - 8\alpha^2\ddot{G}_2 + 16\alpha^4 G_2 + 2i\alpha R[(1-y^2-c_{1r})(\ddot{G}_2 - 4\alpha^2 G_2) + 2G_2] = i\alpha R(\phi_1\dddot{\phi}_1 - \dot{\phi}_1\ddot{\phi}_1). \tag{24}$$

Here, c_{1r} denotes the real part of the eigenvalue of (17), and in the approximations use was made of the relation

$$dA_1/dt = i\alpha c_{1r}A_1 + O(\epsilon^2). \tag{25}$$

With the right-hand side of (24) a known odd function of y, G_2 is the odd solution satisfying boundary conditions like those for ϕ_1 given in (18).

Equation (12) takes on the form

$$M(f_1) \equiv \left\{ f_i^{\mathrm{IV}} - 2\alpha^2 \ddot{f}_1 + \alpha^4 f_1 + i\alpha R[(1-y^2)(\ddot{f}_1 - \alpha^2 f_1) + 2f_1] - R\frac{\partial}{\partial t}(\ddot{f}_1 - \alpha^2 f_1) \right\}$$
$$= \epsilon^3 |A_1|^2 A_1 RP(y), \qquad (26)$$

with $P(y) = -i\alpha[\dot{G}_0(\ddot{\phi}_1 - \alpha^2\phi_1) - \dddot{G}_0\phi_1 + \bar{\phi}_1 \dddot{G}_2$
$$- 3\alpha^2 \bar{\phi}_1 \dot{G}_2 - \ddot{\bar{\phi}}_1 \dot{G}_2 - 2\dddot{\bar{\phi}}_1 G_2 - 6\alpha^2 \dot{\bar{\phi}}_1 G_2 + 2\dot{\bar{\phi}}_1 \ddot{G}_2]. \qquad (27)$$

Substituting the expansion (16) into (26), we get, by virtue of (17),

$$i\alpha Rc_1(\ddot{\phi}_1 - \alpha^2\phi_1)A_1 - R(\ddot{\phi}_1 - \alpha^2\phi_1)(dA_1/dt) + \sum_{n=2}^{\infty} A_n M(\phi_n) = \epsilon^2 |A_1|^2 A_1 RP(y). \qquad (28)$$

We now consider the system adjoint to (17):

$$\tilde{\phi}_n^{\mathrm{IV}} - 2\alpha^2 \ddot{\tilde{\phi}}_n + \alpha^4 \tilde{\phi}_n + i\alpha R[(1 - y^2 - \bar{c}_n)(\ddot{\tilde{\phi}}_n - \alpha^2\tilde{\phi}_n) - 4y\dot{\tilde{\phi}}_n] = 0, \qquad (29)$$
$$\tilde{\phi}_n(1) = \dot{\tilde{\phi}}_n(1) = \tilde{\phi}_n(-1) = \dot{\tilde{\phi}}_n(-1) = 0. \qquad (30)$$

It can be shown (Eckhaus 1965) that the following orthogonality relations hold:

$$\int_{-1}^{1} (\ddot{\phi}_k - \alpha^2\phi_k)\tilde{\phi}_n \, dy = \delta_{kn}, \qquad (31)$$

$$\int_{-1}^{1} \sum_{n=2}^{\infty} A_n M(\phi_n)\tilde{\phi}_1 \, dy = 0. \qquad (32)$$

Hence, by multiplying (28) by $\tilde{\phi}_1$ and integrating from $y = -1$ to $y = 1$ we get

$$\frac{dA_1}{dt} = i\alpha c_1 A_1 - 2\epsilon^2 |A_1|^2 A_1 \beta, \qquad (33)$$

where
$$\beta(\alpha, R) = \frac{1}{2}\int_{-1}^{1} P(y)\tilde{\phi}_1(y)\, dy = \beta_r + i\beta_i. \qquad (34)$$

If we take the complex conjugate of (33)

$$\frac{d\bar{A}_1}{dt} = -i\alpha\bar{c}_1\bar{A}_1 - 2\epsilon^2|A_1|^2\bar{A}_1\bar{\beta}, \qquad (35)$$

multiply (35) by A_1, (33) by $\bar{A}_1$, and add, we get

$$\frac{d|A_1|^2}{dt} = -2\alpha c_{1i}|A_1|^2 - 4\epsilon^2|A_1|^4\beta_r, \qquad (36)$$

where β_r denotes the real part of β.

For $R > R_c$, c_{1i} is negative, so that if β_r is positive, (36) can have a solution for which

$$\frac{d|A_1|^2}{dt} \to 0, \quad t \to \infty, \qquad (37)$$

$$|A_1|^2 \to -\frac{\alpha c_{1i}}{2\epsilon^2\beta_r} = -\frac{c_{1i}}{2|c_{1i}|\beta_r}. \qquad (38)$$

On the other hand, if β_r is negative, no such periodic disturbance of finite amplitude is possible in the supercritical region.

We have carried out the calculation and found that there indeed exists a zone in the (α, R)-plane where β_r is positive, as shown in figure 1.

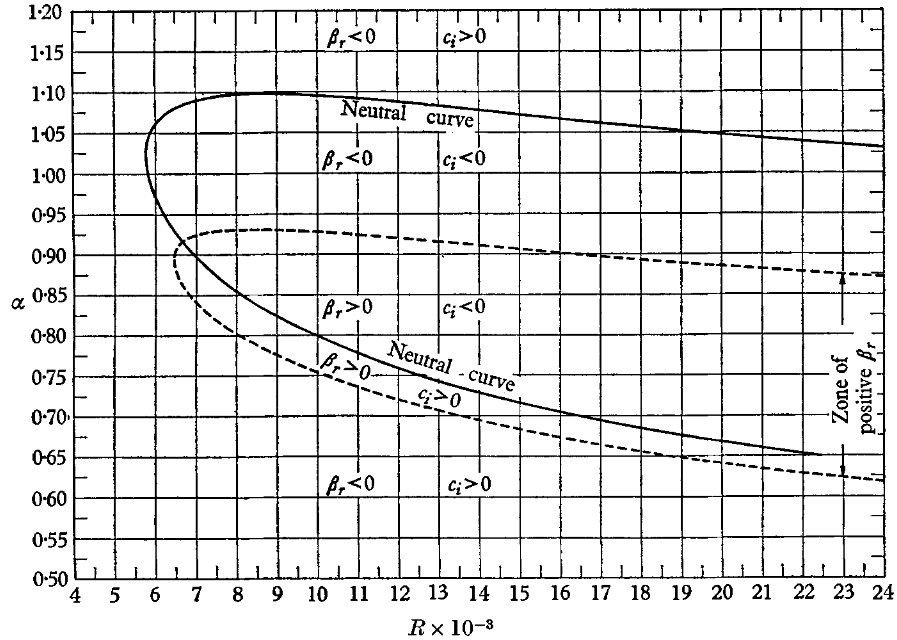

FIGURE 1. β_r of equation (34) is positive inside the region bounded by the dashed curve. The zone inside the neutral curve is unstable in the linear theory.

3. Method of Solution

In order to evaluate the integral β in (34), one has to solve (29) for $\tilde{\phi}_1 (n = 1)$ and also (17) and (24) for the functions ϕ_1 and G_2 which enter into $P(y)$, given in (27). We shall indicate here the method we used for the solution of (24) for G_2. Assuming that we have solved (17) for ϕ_1, so that the right-hand side of (24), $K(y)$, is known, where

$$K(y) \equiv i\alpha R(\phi_1 \ddot{\tilde{\phi}}_1 - \tilde{\phi}_1 \ddot{\phi}_1), \tag{39}$$

one can solve (24) by using the Runge–Kutta method, or Nordsieck's method. We have found that even more effective is the method used by Thomas (1953). Thomas introduces an auxiliary function $\Gamma(y)$ defined by

$$\Gamma(y) = G_2 - \frac{h^2}{6}\ddot{G}_2 + \frac{h^4}{90}G^{\text{IV}}, \tag{40}$$

where h denotes the interval of integration. The coefficients in (40) were so chosen as to yield the relations

$$G_2 = \Gamma + \tfrac{1}{6}\delta^2\Gamma + \tfrac{1}{360}\delta^4\Gamma - \left[\frac{67h^8}{907{,}200}G_2^{(8)}\right], \tag{41}$$

$$\ddot{G}_2 = \frac{1}{h^2}(\delta^2\Gamma + \tfrac{1}{12}\delta^4\Gamma) - \left[\frac{h^6}{6048}G_2^{(8)}\right], \tag{42}$$

$$G_2^{\text{IV}} = \frac{1}{h^4}\delta^4\Gamma + \left[\frac{h^4}{240}G_2^{(8)}\right]. \tag{43}$$

Here the δ's denote central differences, and the terms in brackets give the truncation error. Substitution of (41)–(43) in (24) at each point

$$y_k = kh, \quad h = 1/n, \tag{44}$$

leads to the recursion relations for the Γ_k

$$P_k\Gamma_{k+2} + T_k\Gamma_{k+1} + W_k\Gamma_k + T_k\Gamma_{k-1} + P_k\Gamma_{k-2} = h^4K_k, \tag{45}$$

where
$$P_k = 1 + \frac{h^2}{12}r_k + \frac{h^4}{360}s_k, \tag{46}$$

$$Q_k = h^2r_k + \frac{h^4}{6}s_k, \tag{47}$$

$$T_k = Q_k - 4P_k; \quad W_k = 6P_k - 2Q_k + h^4s_k, \tag{48}$$

$$r_k = -8\alpha^2 + 2i\alpha R(1 - h^2k^2 - c), \tag{49}$$

$$s_k = 16\alpha^4 + 4i\alpha R - 8i\alpha^3R(1 - h^2k^2 - c). \tag{50}$$

Equation (45) takes on special forms near the centre of the channel ($k = 0, 1$) and near the wall ($k = n-1, n$). Because $\Gamma(y)$ is an odd function of y, we have

$$\Gamma_{-1} = \Gamma_1, \quad \Gamma_{-2} = \Gamma_2, \tag{51}$$

whereby the first two equations in (45) become

$$P_1\Gamma_3 + T_1\Gamma_2 + (W_1 - P_1)\Gamma_1 = h^4K_1 \quad (k = 1), \tag{52}$$

$$P_2\Gamma_4 + T_2\Gamma_3 + W_2\Gamma_2 + T_2\Gamma_1 = h^4K_2 \quad (k = 2). \tag{53}$$

Near the wall, the condition $G_2(1) = 0$ gives, by (41),

$$360G_2(1) = \Gamma_{n+2} + 54\Gamma_{n+1} + 246\Gamma_n + 54\Gamma_{n-1} + \Gamma_{n-2} = 0. \tag{54}$$

For the condition $\dot{G}_2(1) = 0$, we use, with Thomas, the less accurate approximation

$$\dot{G}_2(1) = \frac{1}{2h}(\Gamma_{n+1} - \Gamma_{n-1}) + \left[\frac{h^4}{120}G_2^{(5)}\right], \tag{55}$$

leading to
$$\Gamma_{n+1} = \Gamma_{n-1} + O(h^5). \tag{56}$$

Equations (54) and (56) can be used to solve for Γ_{n+1} and Γ_{n+2}, whereby the last two equations in (45) become

$$T_{n-1}\Gamma_n + (W_{n-1} + P_{n-1})\Gamma_{n-1} + T_{n-1}\Gamma_{n-2} + P_{n-1}\Gamma_{n-3} = h^4K_{n-1} \quad (k = n-1), \tag{57}$$

$$(W_n - 246P_n)\Gamma_n + (2T_n - 112P_n)\Gamma_{n-1} = 0 \quad (k = n). \tag{58}$$

The system of equations (45), with the special cases (52), (53), (57) and (58), were solved by Gaussian elimination using an integration interval $h = 0{\cdot}01$ and occasional checks with $h = 0{\cdot}005$.

4. Discussion of Results

Before giving the results for β, it is important to specify the normalization conventions which we have adopted. The solution ϕ_1 of (17) was normalized by putting $\phi_1(0) = 1$; the normalization of the adjoint eigenfunction $\tilde{\phi}_1$ of equation (29) (with $n = 1$) was made according to (31).

R ...	4000		4500		5000		6000		8000		10,000	
α	β_r	β_i	β_r	β_i	β_r	β_i	β_r	β_i	β_r	β_i	β_r	β_i
1·20	−143·7	−256·7	−167·3	−287·5	−192·6	−319·8	−248·0	−390·3	−377·0	−558·9	−529·6	−770·5
1·15	−92·2	−201·0	−106·7	−222·1	−122·4	−243·9	−157·1	−289·7	−238·4	−394·4	−334·7	−521·4
1·10	−56·1	−159·3	−64·0	−174·5	−72·9	−189·7	−93·3	−220·7	−142·4	−287·2	−201·1	−363·4
1·05	−32·5	−126·7	−35·5	−138·0	−39·3	−149·2	−49·4	−171·2	−76·6	−215·6	−110·6	−262·8
1·00	−19·3	−100·7	−18·7	−109·1	−18·9	−117·3	−21·4	−133·6	−32·7	−165·1	−50·0	−196·3
0·95	−14·2	−80·2	−11·5	−86·1	−9·3	−92·1	−6·5	−103·9	−6·2	−126·9	−11·4	−149·0
0·90	−14·8	−64·8	−11·2	−68·6	−8·0	−72·7	−2·3	−80·7	5·6	−96·8	9·0	−112·7
0·85	−18·3	−53·8	−14·9	−56·3	−11·6	−58·8	−5·1	−63·8	6·0	−74·1	14·5	−84·7
0·80	−22·7	−46·0	−20·0	−47·7	−17·3	−49·2	−11·6	−52·3	−0·5	−58·3	9·5	−64·6
0·75	−26·2	−40·4	−24·6	−41·6	−22·8	−42·7	−18·7	−44·7	−9·7	−48·2	−0·7	−51·6
0·70	−27·8	−35·8	−27·4	−37·0	−26·7	−37·9	−24·5	−39·5	−18·6	−41·8	−11·9	−43·7
0·65	−27·0	−31·4	−27·5	−32·7	−27·8	−33·8	−27·6	−35·4	−25·2	−37·5	−21·3	−38·9
0·60	−23·8	−26·3	−25·0	−27·9	−26·0	−29·2	−27·3	−31·3	−28·1	−33·9	−27·1	−35·5
0·55	−19·2	−20·5	−20·5	−22·2	−21·7	−23·6	−23·8	−26·1	−26·7	−29·6	−28·2	−31·9
0·50	−14·6	−14·5	−15·7	−16·0	−16·7	−17·3	−18·7	−19·8	−22·0	−23·7	−24·6	−26·7

TABLE 1. The integral $\beta(\alpha, R) = \beta_r + i\beta_i$ defined in equation (34)

R ...	12,000		14,000		16,000		18,000		20,000		24,000	
α	β_r	β_i	β_r	β_i	β_r	β_i	β_r	β_i	β_r	β_i	β_r	β_i
1·20	−704·1	−1031·0	−897·3	−1344·0	−1105·3	−1711·4	−1323·0	−2132·8	−1545·2	−2605·0	−1985·1	−3678·0
1·15	−445·8	−674·8	−571·0	−857·9	−709·4	−1073·6	−859·4	−1324·0	−1018·9	−1610·5	−1355·2	−2292·3
1·10	−268·8	−452·4	−345·4	−556·6	−431·0	−677·7	−525·3	−817·5	−628·0	−977·6	−855·7	−1363·6
1·05	−150·2	−314·9	−195·1	−373·8	−245·3	−440·5	−300·8	−516·3	−361·7	−601·9	−499·4	−806·3
1·00	−71·5	−228·3	−96·5	−262·5	−124·6	−299·7	−155·7	−340·6	−189·8	−385·7	−267·1	−490·9
0·95	−20·6	−170·5	−32·7	−192·0	−47·0	−213·9	−63·3	−236·9	−81·3	−261·3	−122·4	−315·5
0·90	8·9	−128·1	6·0	−143·0	0·9	−157·5	−5·8	−171·9	−14·0	−186·3	−33·6	−216·1
0·85	20·4	−95·3	24·1	−105·8	25·8	−116·1	25·8	−126·1	24·5	−135·8	18·6	−154·6
0·80	18·2	−71·1	25·4	−77·7	31·3	−84·5	35·9	−91·2	39·4	−97·9	43·4	−111·0
0·75	7·9	−55·0	15·9	−58·7	23·2	−62·4	29·8	−66·3	35·7	−70·3	45·5	−78·5
0·70	−5·0	−45·5	1·9	−47·2	8·5	−49·0	14·9	−50·9	21·0	−52·8	32·3	−56·9
0·65	−16·7	−40·0	−11·9	−40·9	−6·9	−41·7	−1·8	−42·5	3·1	−43·3	12·8	−45·0
0·60	−25·1	−36·5	−22·5	−37·2	−19·6	−37·8	−16·4	−38·2	−13·0	−38·6	−6·2	−39·3
0·55	−28·6	−33·4	−28·3	−34·4	−27·4	−35·1	−26·1	−35·6	−24·6	−36·0	−20·9	−36·5
0·50	−26·6	−28·9	−28·0	−30·6	−28·9	−31·8	−29·4	−32·8	−29·5	−33·5	−29·0	−34·5

TABLE 2. The integral $\beta(\alpha, R) = \beta_r + i\beta_i$ defined in equation (34)

The function $\beta(\alpha, R)$ given in (34) was mapped out in the region $0{\cdot}5 < \alpha < 1{\cdot}2$, $4000 < R < 24{,}000$, and the results are shown in tables 1 and 2. Since the integrand is a rapidly varying function near the wall, we have repeated the calculation with the integrand evaluated at 200 points, instead of 100. The results agreed to about 1 part in 1000 for $|\beta|$. Equally, the use of $(c_{1r} + c_{1i})$ in (24), in place of c_{1r}, did not change $|\beta|$ appreciably.

The interesting feature shown in tables 1 and 2 is that there exists a zone in the (α, R)-plane where the real part of β is positive. This zone is located inside the dashed curve of figure 1. The asymptotic periodic solutions of finite amplitude predicted by Stuart for the supercritical region could therefore exist inside the dashed zone, just above the neutral curve. It would also follow that outside the dashed curve the values of β_r (negative) given in table 1 could be used to determine the displacement of the neutral curve towards smaller Reynolds numbers as a result of the finiteness of the amplitude of perturbation. This question will be dealt with in a forthcoming communication.

It is well to keep in mind that Stuart's theory is based on certain order-of-magnitude estimates given in equations (19), (20), (21), and on the neglect of the term $R\partial \mathring{f}_0/\partial t$ in (15). The terms in (6) above $n = 2$ were also stated to be negligible. To the extent that these estimates are valid—and Eckhaus' (1965) analysis supports them—the existence of an overlapping region where $\beta_r > 0$, and $c_i < 0$, that we have found, lends support to Stuart's deduction of the existence of a periodic disturbance of the plane Poiseuille flow which grows to a finite asymptotic amplitude, as given in equations (37) and (38).

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